factor in the first formula of (9), i.e. the perturbations fade with time. Hence the rotation of a viscous incompressible fluid is stable with respect to such perturbations.

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FLOW AROUND A SPHERICAL DROP AT INTERMEDIATE REYNOLDS NUMBERS

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A solution of the Navier-Stokes equations for the flow of fluid in and outside a drop with conditions of matching at the interface is derived by the method of finite differences. Drag coefficients are determined in the range $(0.5 \leqslant \text{Re} \leqslant 100)$ of Reynolds numbers for a solid sphere, a drop, and a small gas bubble. Vortex and velocity distribution at the drop boundary is determined.

The flow around a solid sphere in the intermediate range of Reynolds numbers had been thoroughly investigated [1]. Solutions for the problem of flow around a spherical drop are presented in [2, 3] for $\text{Re} \ll 1$. The method of joining asymptotic expansions was used in [4] for obtaining a solution for small Re with allowance for inertia terms in the Navier-Stokes equations. In [5 — 7] solutions were derived for $\text{Re} \gg 1$ in the boundary layer approximation (a detailed analysis of approximate solutions for low and high Re appeared in the survey paper [8]). The particular case of the drop of water in air, which is distinguished by the high ratio viscosities of the inner and outer media ($\mu \approx 56$), was investigated in [9] in the intermediate range of Reynolds numbers by the method of finite differences. It was shown there that for such μ the drag of the moving drop is virtually the same as that of the solid sphere. Here, the drag of the drop is investigated for $0 \leqslant \mu < \infty$ and $\text{Re} \leqslant 100$.

The rectilinear uniform motion of a drop in a homogeneous mass force field is considered. The Weber number is assumed to be fairly low so that the drop virtually retains its spherical shape.

In a system of coordinates attached to the drop the motion is steady and axisymmetric up to $\text{Re} \approx 100$, as in the case of a solid sphere [10].

With the coordinate origin located at the drop center and the polar axis directed downstream $(\theta = 0)$, the Navier-Stokes equations for the fluid flow in and outside the drop and the boundary conditions at the drop surface, expressed in terms of variables ψ (the stream function) and ζ (the vortex), are of the form

$$\left[\frac{\partial\psi_i}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{\zeta_i}{r\sin\theta}\right)-\frac{\partial\psi_i}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{\zeta_i}{r\sin\theta}\right)\right]\sin\theta=\frac{2}{\mathrm{Re}_i}E^2(\zeta_i r\sin\theta) \tag{1}$$

$$E^2\psi_i+\zeta_i r\sin\theta=0 \quad (i=1,\,2)$$

$$E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right), \qquad \mathrm{Re}_i=\frac{2u_\infty a\rho_i}{\mu_i}$$

$$r=1,\quad v_{r1}=v_{r2}=0,\quad v_{\theta1}=v_{\theta2},\quad \mu(\zeta_1-2v_{\theta1})=\zeta_2-2v_{\theta2}\quad (\mu=\mu_1/\mu_1)$$

where Re_i are Reynolds numbers; ρ_i are densities, and μ_i are dynamic viscosities, with subscripts $i=1$ and 2 denoting quantities in the inner and outer regions, respectively. The velocity u_∞ away from the drop and the drop radius a are taken as units of velocity and length, respectively.

Besides the specified above conditions of continuity of the velocity vector and of the tension tensor at the drop surface, conditions of axial symmetry and steadiness of the flow at infinity must be satisfied.

For solving the problem in the outer region we introduce the new unknown ψ_2^* related to ψ_2 by the formula $\psi_2=\frac{1}{2}r^2\sin\theta+\psi_2^*$, and transform the outer region into a semicircle by the substitution $r=1/\eta$.

The method of variable directions is used. The first derivatives are approximated by central differences of second order for both the constant and the variable pitch grids. Here, unlike in [11], the difference analog of joining conditions represented by the last two equalities in (1) is satisfied as follows. At each n-th layer the new ζ_1^n and ζ_2^n at a particular point of the boundary are determined by the values of ζ_1^{n-1}, ζ_2^{n-1}, $v_{\theta1}^{n-1}$ and $v_{\theta2}^{n-1}$ at the corresponding points of the preceding layer with the use of formulas

for $\mu\leqslant1$

$$\zeta_1^n=\zeta_1^{n-1}+\beta(v_{\theta2}^{n-1}-v_{\theta1}^{n-1})$$

$$\zeta_2^n=\mu\zeta_1^n+(1-\mu)(v_{\theta1}^{n-1}+v_{\theta2}^{n-1})$$

for $\mu>1$

$$\zeta_2^n=\zeta_2^{n-1}+\beta(v_{\theta2}^{n-1}-v_{\theta1}^{n-1})$$

$$\zeta_1^n=\frac{\zeta_2^n}{\eta}+\frac{\mu-1}{\mu}(v_{\theta1}^{n-1}+v_{\theta2}^{n-1})$$

where β is a parameter determined by the condition of best convergence ($0.1\lesssim\beta\lesssim1$).

This method of approximating boundary conditions for fluid-fluid interface is an extension of Dorodnitsyn's method (for solid body-fluid interface). It becomes identical to the latter for $\mu\to\infty$.

The computation of a single variant for each of the two regions using a 20×20 grid requires 5— 10 minutes of computer time.

The controlling parameters of the problem are: Re_2, μ_1/μ_2 and ρ_1/ρ_2. The last of these appears in the system of equations of our problem in terms of $\mathrm{Re}_1=(\rho_1/\rho_2)(\mu_2/\mu_1)\mathrm{Re}_2$. The effect of Re_1 on the flow was specifically investigated. As shown in [10], the inner flow for $\mathrm{Re}_2\ll1$ is defined by Hill's vortex. Since the latter reduces separately the convection and viscosity terms to zero, it represents the exact solution of the Navier-Stokes equations, which is independent of the Re number. Hence at small Re_2 the Re_1 number does not affect the stream. Computations how that for fixed μ and Re_2 in the range $1<\mathrm{Re}_2<100$ the variation of Re_1 within the range $1<\mathrm{Re}_1<100$ has virtually no effect on the flow characteristics (variation of the drag coefficient is of the order of 1%). In our computations we assumed $Re_1 = Re_2$.

Table 1

$\mu \setminus Re_2$	0.5	1	2	5	10	20	50	100
0	33.8	17.5	9.3	4.25	2.43	1.41	0.69	0.40
0.333	38.3	19.9	10.6	4.89	2.87	1.71	0.89	0.55
1	42.9	22.4	12.1	5.65	3.33	2.05	1.12	0.74
3	47.4	24.8	13.4	6.36	3.80	2.38	1.36	0.96
∞	52:2	27.4	14.7	7.05	4.28	2.71	1.58	1.11

Drag coefficients of the solid ball ($\mu = \infty$), gas bubble ($\mu = 0$) and the drop ($\mu = 0.333$, 1, 3) are specified in Table 1. The drag coefficient of the solid sphere varies not more than 1% from that calculated in [1] which shows a good correlation with experimental data (a 100×60 grid was used in [1]).

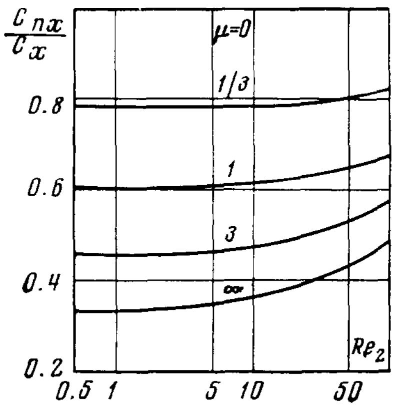

Fig. 1

Ratio of the drag coefficient of a gas bubble to that of a solid sphere is for $Re_2 \ll 1$ equal to $^2/_3$. Table 1 shows that this ratio decreases with increasing Re_2, reaching 0.366 for $Re_2 = 100$. It may be assumed that the character of dependence of the drag coefficient on the ratio of viscosities remains unchanged also for $Re_2 > 1$ Hence, by analogy to the case of $Re_2 \ll 1$, the drag coefficient of the drop can be defined in terms of drag coefficients of the solid ball (C_{x_∞}) and of the gas bubble (C_{x_0}) for the same Re_2 by the formula

$$C_x \approx \frac{\mu C_{x_\infty} + C_{x_0}}{\mu + 1}$$

This formula yields values of the drag coefficient that coincide with calculated values (see Table 1) with an error not exceeding 5%, hence it can be used in practical computations. The dependence of the ratio of the form drag coefficient to that of total drag (C_{nx} / C_x), i.e. the contribution of the normal component of tension to total drag, on Re_2 is shown in Fig. 1. Low μ contributes to greater mobility of the boundary and a more intensive circulation of fluid inside the drop. This results in the decrease of the contribution of tangential tension and in the increase of the ratio C_{nx} / C_x which reaches its maximum for $\mu = 0$ (tangential tension at the bubble surface is zero and the total drag coefficient is equal to that of form drag). The ratio C_{nx} / C_x in the case of poorly streamlined bodies (solid sphere, cylinder) increases with increasing Re. This is due to the appearance of a stagnation zone followed by flow separation. This increase becomes less pronounced with decreasing μ, owing to the slower development of the stagnation and separation zones.

Vortex and velocity distribution at the surface of a sphere are shown in Fig.2 for several values of Re_2 and μ (angle θ is measured from the leading stagnation point). Curves 1 relate to $Re_2 = 0.5$ and $\mu = 1$; curves 2 and 3 relate to $Re_2 = 20$ and $\mu = 1$ and ∞; curves 4, 5, 6 and 7 relate to $Re_2 = 100$ and $\mu = 0$, 1, 10 and ∞, respectively.

The change of μ from ∞ to 0 (as applicable to the solid ball, the drop and gas bubble) is accompanied for a fixed Re_2 by a reduction of the vortex and simultaneous increase of velocity at the surface, with a widening of the range of these parameters. Such change (of μ) results in the decrease of tangential tension at the sphere surface which can be

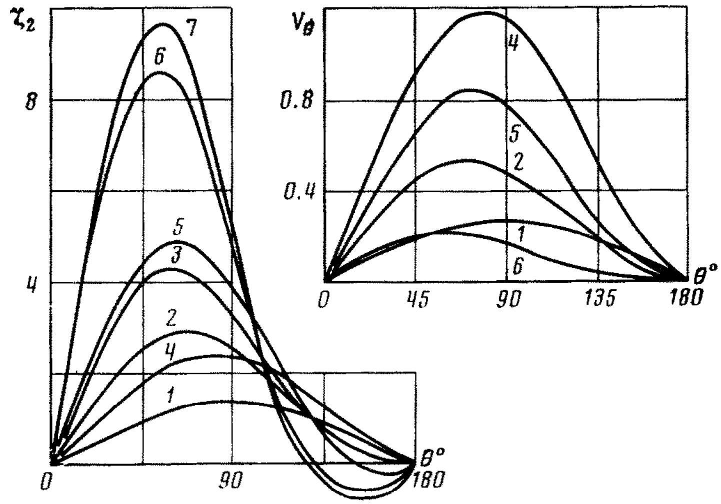

Fig. 2

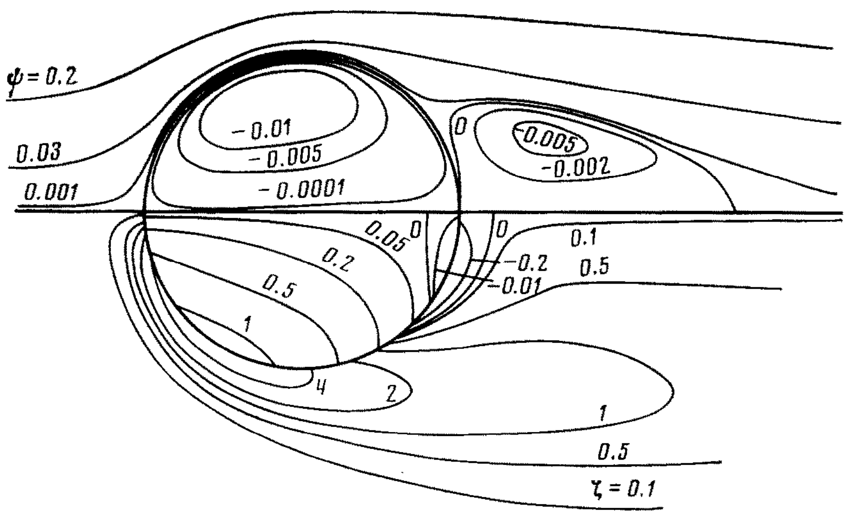

Fig. 3

expressed in terms of the vortex and velocity by the formula

$$\tau_{r\theta} = \frac{4}{\mathrm{Re}_2} (\zeta_2 - 2v_\theta)_{r=1}$$

The asymmetric distribution of tangential forces over the sphere surface becomes more pronounced with increasing Re_2 and μ . In spite of this the pattern of streamlines inside the drop (Fig. 3) is only slightly different from that of a flow corresponding to Hill's vortex even for relatively high Re_2 and μ, while the vortex intensity distribution differs considerably (in the case of Hill's vortex lines $\zeta = \text{const}$ are straight lines parallel to the polar axis). Streamlines and the vortex distribution (lines $\zeta = \text{const}$) for $Re_2 = 100$ and $\mu = 10$ are shown in Fig. 3 (this variant was computed with the use of a 40×40 grid in each of the two regions). Owing to the nonzero velocity at the interface of the two fluids, the separation point of the zero streamline is determined by the vanishing of velocity and not by the tangential tension, as is the case with the flow around a solid body. That point does not determine the reverse flow zone, whose size can be defined by the angle of maximum spread of the attached vortex in the upstream direction (the angle between the polar axis and the tangent to the zero streamline drawn from the coordinate origin).

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